On the Regularity of the Landweber Transform

Andrzej Cegielski (Zielona Góra, Poland)

A joint work with Simeon Reich and Rafał Zalas

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Outline

Nonexpansive and Quasi-Nonexpansive Operators

Definition

Let $\mathcal{H}$ be a Hilbert space. We say that an operator $T : \mathcal{H} \to \mathcal{H}$ is:

nonexpansive (NE), if

$$\forall_{x,y\in\mathcal{H}} \quad \|Tx - Ty\| \leq \|x - y\|\,;$$

α-averaged, $\alpha \in (0,1)$, if $T = \alpha S + (1-\alpha)\,\mathrm{Id}$ for a NE operator S;

firmly nonexpansive (FNE), if T is $\frac{1}{2}$-averaged;

quasi-nonexpansive (QNE), if

$$\forall_{x\in\mathcal{H}, z\in\mathrm{Fix}\,T\neq\emptyset} \quad \|Tx - z\| \leq \|x - z\|\,;$$

ρ-strongly quasi-nonexpansive (ρ-SQNE), *where* $\rho > 0$, if

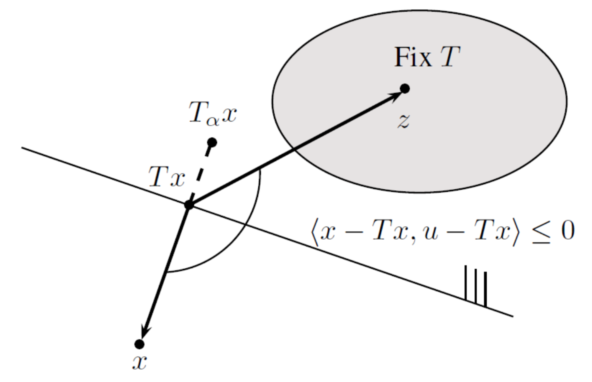
$$\forall_{x\in\mathcal{H}, z\in\mathrm{Fix}\,T\neq\emptyset} \quad \|Tx - z\|^2 \leq \|x - z\|^2 - \rho\|Tx - x\|^2;$$

a *cutter* if

$$\forall_{x\in\mathcal{H}, z\in\mathrm{Fix}\,T\neq\emptyset} \quad \langle z - Tx, x - Tx\rangle \leq 0;$$

($\Leftrightarrow$ T is 1-SQNE).

Nonexpansive and Quasi-Nonexpansive Operators



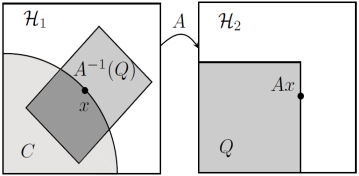
cutter and SQNE operator

Split Feasibility Problem and the Landweber Transform

$\mathcal{H}_1$ and $\mathcal{H}_2$ - two real Hilbert spaces, $A : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded linear operator. The *Split Feasibility Problem* (SFP) is to

$$\text{find } x \in C \text{ such that } Ax \in Q,$$

where $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ are closed and convex.



SFP with $\mathcal{H}_1 = \mathbb{R}^n$ and $\mathcal{H}_2 = \mathbb{R}^m$ was introduced by Censor and Elfving in 1994.

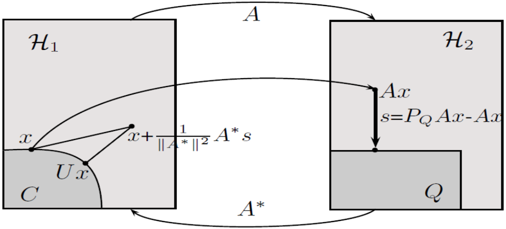
Split Feasibility Problem and the LandweberTransform

In 2002 Byrne proposed the following *CQ-method* (in a finite dimensional case)

$$x^{k+1} = P_C(x^k + \frac{\lambda}{\|A\|^2} A^T(P_Q A x^k - A x^k)),$$

where $\lambda \in (0,2)$ and $\|A\|$ is the spectral norm of A, and proved its convergence to

$$\operatorname*{Argmin}_{x \in C} \frac{1}{2} \|P_Q(Ax) - Ax\|^2 \qquad (= \mathrm{Fix}(\mathrm{Id} + A^T(P_Q - \mathrm{Id})A)).$$



Split Feasibility Problem and the Landweber Transform

If $Q = \{b\} \subseteq \mathcal{H}_2$ then the SFP is to find $x \in C$ with $Ax = b$. In this case the CQ-method is called the *projected Landweber method*.
Replace P_C and P_Q in the CQ-method by sequences of QNE operators S_k and T_k and the parameter λ by a sequence $\lambda_k \in (0, 1)$

$$x^{k+1} = S_k(x^k + \frac{\lambda_k}{\|A\|^2} A^*(T_k(Ax^k) - Ax^k)). \tag{1}$$

We introduce a *Landweber transform* denoted by $\mathcal{L}_A\{\cdot\}$ or shortly $\mathcal{L}\{\cdot\}$, which for a given operator $T : \mathcal{H}_2 \to \mathcal{H}_2$ assigns an operator $\mathcal{L}\{T\} : \mathcal{H}_1 \to \mathcal{H}_1$ defined by

$$\mathcal{L}_A\{T\} := \mathrm{Id} + \frac{1}{\|A\|^2} A^*(T - \mathrm{Id})A$$

which we call a *Landweber operator* (related to T).
Denoting $U_\lambda := \mathrm{Id} + \lambda(U - \mathrm{Id})$ – the λ-relaxation of U, we write (1) as:

$$x^{k+1} = S_k \mathcal{L}\{(T_k)_{\lambda_k}\}$$

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3. C.–Reich–Zalas, 2020: Conditions for the weak / strong /linear convergence.

Regular Families of Sets and Regular Operators

Definition (Bauschke–Borwein, 1996)

- Let $\mathcal{C}$ be a family of closed convex subsets $C_i \subseteq \mathcal{H}$, $i \in I := \{1, 2, ..., m\}$, with $C := \bigcap_{i \in I} C_i \neq \emptyset$. We say that $\mathcal{C}$ is:

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  - *(boundedly) regular* if for any (bounded) sequence $\{x^k\}_{k=0}^{\infty}$

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  - (*boundedly*) *linearly regular* if (for any bounded $D \subseteq \mathcal{H}$) there is a constant $\kappa > 0$ such that for every $x (\in D)$
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  - If all C_i, $i \in I$, are half-spaces, then $\mathcal{C}$ is linearly regular;

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- The metric projection P_C is linearly regular.

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  - *If U is boundedly linearly regular, then the convergence is linear.*

- The theorem is also true if we replace U by a sequence of operators U_k. In this case one should define a (weakly, linearly) regular sequence operators.

Properties of the Landweber Operator

$$\mathcal{L}\{T\} := \mathrm{Id} + \frac{1}{\|A\|^2} A^*(T - \mathrm{Id})A$$

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  - In particular, if T is a cutter then $\mathcal{L}\{T\}$ is also a cutter.

Theorem

- *Let* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *be nonzero bounded linear,* $T : \mathcal{H}_2 \to \mathcal{H}_2$ *be QNE with* $\operatorname{im} A \cap \operatorname{Fix} T \neq \emptyset$*, and* $\mathcal{L}\{T\} : \mathcal{H}_1 \to \mathcal{H}_1$ *be the Ladweber operator, defined by*

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Extrapolated Landweber Operator

Let $T : \mathcal{H}_2 \to \mathcal{H}_2$, and $\sigma : \mathcal{H}_1 \to [1, \infty)$ be an *extrapolation function*.

Definition

The operator $\mathcal{L}_\sigma\{T\} : \mathcal{H}_1 \to \mathcal{H}_1$, defined by

$$\mathcal{L}_\sigma\{T\}x := x + \sigma(x)(\mathcal{L}\{T\}x - x), \tag{2}$$

is called an *extrapolated Landweber operator* (related to σ).

Theorem (C., 2016, C.–Reich–Zalas, 2020)

Let T be QNE, Fix $\mathcal{L}\{T\} \neq \emptyset$ *and the extrapolation function σ satisfies*

$$\sigma(x) \leq \frac{\|A\|^2 \cdot \|T(Ax) - Ax\|^2}{\|A^*(T(Ax) - Ax)\|^2} \text{ for all } x \in \mathcal{H}_1 \tag{3}$$

Then $\mathcal{L}_\sigma\{T\}$ is QNE. Conditions for weak/strong/linear regularity of $\mathcal{L}_\sigma\{T\}$ are similar to those of $\mathcal{L}\{T\}$.

Projected Extrapolated Landweber Method

Let S, T be QNE, $F := \operatorname{Fix} S \cap \operatorname{Fix} \mathcal{L}\{T\} \neq \emptyset$,

$$x_0 \in \mathcal{H}_1, \text{ and } x_{k+1} = S_{\mu_k}\left(x_k + \lambda_k \frac{\sigma(x_k)}{\|A\|^2} A^* T(Ax_k) - Ax_k\right), \qquad (4)$$

where $\mu_k, \lambda_k \in [\varepsilon, 1-\varepsilon]$ for some small $\varepsilon > 0$ and σ satisfies

$$1 \leq \sigma(x) \leq \frac{\|A\|^2 \cdot \|T(Ax) - Ax\|^2}{\|A^*(T(Ax) - Ax)\|^2} \text{ for all } x \in \mathcal{H}_1.$$

Theorem (C.–Reich–Zalas, 2020)

Let x^k be given by (4).

- *If S and T are both weakly regular, then $x_k \rightharpoonup x^* \in F$.*

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  - *If S, T are boundedly regular, and the families $\{\operatorname{im} A, \operatorname{Fix} T\}, \{\operatorname{Fix} S, \operatorname{Fix} \mathcal{L}\{T\}\}$ are boundedly regular, then $x_k \to x$.*

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  - *If S and T are boundedly linearly regular, and the families $\{\operatorname{im} A, \operatorname{Fix} T\}$ and $\{\operatorname{Fix} S, \operatorname{Fix} \mathcal{L}\{T\}\}$ are boundedly linearly regular, then the convergence is linear.*

Linear Split Feasibility Problem

$$\text{Find } x \in \mathbb{R}^n \text{ with } Ax \in Q$$

where A is an $m \times n$ matrix with nonzero rows $a_i \in \mathbb{R}^n$ and $Q := \{y \in \mathbb{R}^m : y \leq b\}$.

- Simultaneous projection for the system $Ax \leq b$

$$P(x) := \sum_{i=1}^{m} w_i (x - \frac{(a_i^T x - b_i)_+}{\|a_i\|^2} a_i,$$

where $w \in \Delta_m := \{v \in \mathbb{R}^m : v_i > 0, i = 1, 2, ..., m$, and $\sum_{i=1}^m v_i = 1\}$.

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- $P(x)$ can be written in a matrix form

$$P(x) = x - A^T D(Ax - b)_+,$$

where $D := WN^{-2}$, $W; = \operatorname{diag} w$, $N := \operatorname{diag}(\|a_1\|, \|a_2\|, ..., \|a_m\|)$. Clearly,

$$D = \operatorname{diag}(\frac{w_1}{\|a_1\|^2}, \frac{w_2}{\|a_2\|^2}, ..., \frac{w_m}{\|a_m\|^2}).$$

Linear Split Feasibility Problem

- The Landweber operator for the SFP has the form

$$\mathcal{L}_A\{P_Q\}(x) = x - \frac{1}{\lambda_{\max}(A^T A)} A^T (Ax - b)_+.$$

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  - $\mathcal{L}_A\{P_Q\}$ is linearly regular
- The system $Ax \leq b$ is equivalent to $D^{\frac{1}{2}} Ax \leq D^{\frac{1}{2}} b$. The simultaneous projection is the same for both systems.
- The Landweber operator for the latter system has the form

$$\mathcal{L}_{D^{\frac{1}{2}} A}\{P_{Q'}\}(x) = x - \frac{1}{\lambda_{\max}(A^T DA)} A^T D(Ax - b)_+,$$

where $Q' := \{y \in \mathbb{R}^m : y \leq D^{\frac{1}{2}} b\}$, and is not equivalent in general to $\mathcal{L}_A\{P_Q\}(x)$.

Linear Split Feasibility Problem

Theorem

- *Let* $w \in \Delta_m$ *and* $I(w) := \{i : w_i > 0\}$. *It holds*

$$\frac{1}{m} \leq w_j := \max\{w_i : i = 1, 2, ..., m\} \leq \lambda_{\max}(A^T DA) \leq 1.$$

Corollary

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    - *If the system* $\{a_i : i \in I(w)\}$ *is orthogonal then* $\lambda_{\max}(A^T DA) = \omega_j$;
    - $\lambda_{\max}(A^T DA) = 1$ *if and only if the system* $\mathcal{A}(w) := \{a_i : i \in I(w)\}$ *is collinear, i.e.,* $a_i = \alpha_i a_1$ *for some* α_i, $i \in I(w)$.

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Corollary

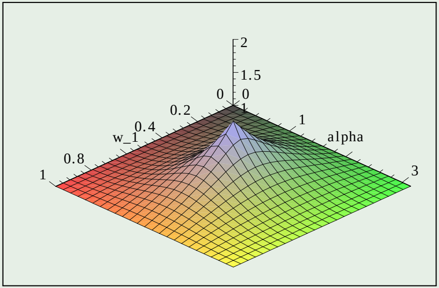
- *The Landweber operator* $\mathcal{L}_{D^{\frac{1}{2}}A}\{P_{Q'}\}$ *is an extrapolation of the simultaneous projection operator* P. *Moreover, if the system* $\mathcal{A}(w)$ *is not collinear then* $\mathcal{L}_{D^{\frac{1}{2}}A}\{P_{Q'}\}$ *is a strict extrapolation of* P.

Linear Split Feasibility Problem

Example

$m = 2$, A has normed rows a_1, a_2 and $w = (w_1, 1 - w_1)$, $\omega_1 \in [0, 1]$, $\alpha = \sphericalangle(a_1, a_2)$.

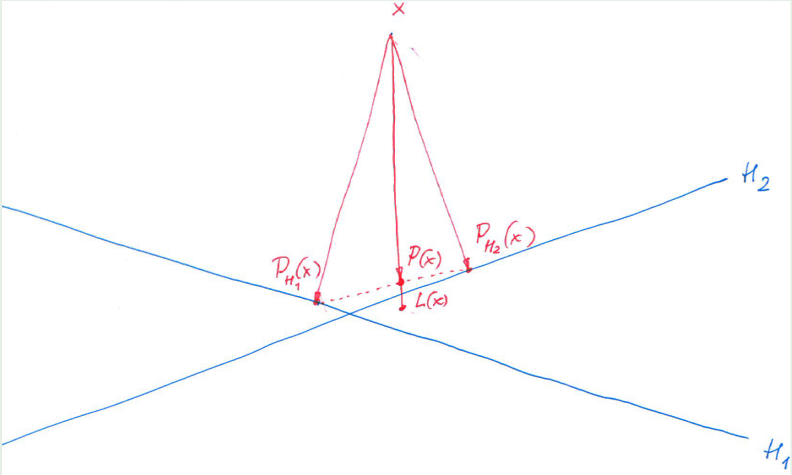
$$\lambda_{\max}(A^T DA) = (1 + \sqrt{1 - 4w_1(1 - w_1)\sin^2\alpha})/2.$$



The extrapolation parameter $\sigma = \frac{1}{\lambda_{\max}(A^T DA)}$ as a function of the weight ω_1 and the angle α

Linear Split Feasibility Problem

Example (continuation)



Linear Split Feasibility Problem

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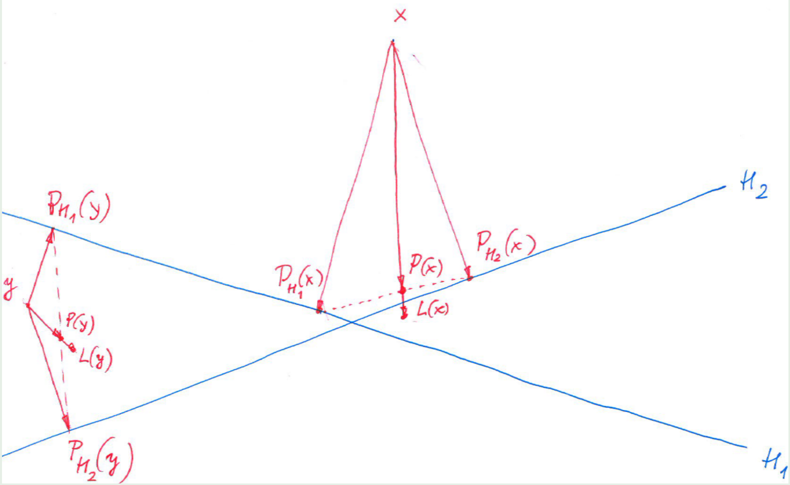
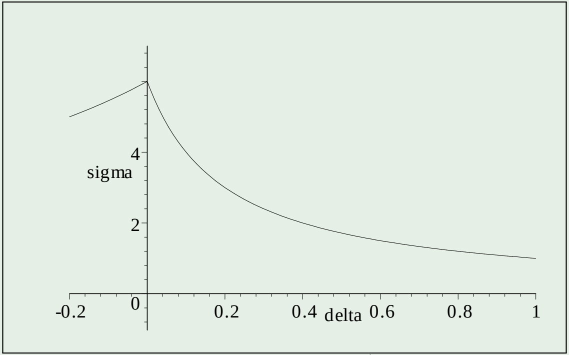


Linear Split Feasibility Problem

Example

A – an $m \times n$ matrix with normed rows a_i, and $\delta := a_i^T a_j$, $i \neq j$, $w_i = \frac{1}{m}$, $i = 1, 2, ..., m$.



The extrapolation parameter $\sigma := \frac{1}{\lambda_{\max}(A^T DA)}$ as a function of δ for $m = 6$

Extrapolated Landweber Operator for Linear Inequalities

- Let the extrapolation function $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ be defined by

$$\sigma(x) = \begin{cases} \frac{(Ax-b)_+^T D(Ax-b)_+}{\|A^T D(Ax-b)_+\|^2} & \text{if } Ax \nleq b \\ 1 & \text{if } Ax \leq b \end{cases}$$

Extrapolated Landweber Operator for Linear Inequalities

- Let the extrapolation function $\sigma : \mathbb{R}^n \to \mathbb{R}_+$ be defined by

$$\sigma(x) = \begin{cases} \frac{(Ax-b)_+^T D(Ax-b)_+}{\|A^T D(Ax-b)_+\|^2} & \text{if } Ax \nleq b \\ 1 & \text{if } Ax \leq b \end{cases}$$

- Then $\sigma(x) \geq \frac{1}{\lambda_{\max}(A^T DA)} \geq 1$ and the operator

$$U_D(x) = x - \sigma(x) A^T D(Ax-b)_+$$

is an extrapolation of the Landweber operator

$$\mathcal{L}_{D^{\frac{1}{2}}A}\{P_{Q'}\}(x) := x - \frac{1}{\lambda_{\max}(A^T DA)} A^T D(Ax-b)_+$$

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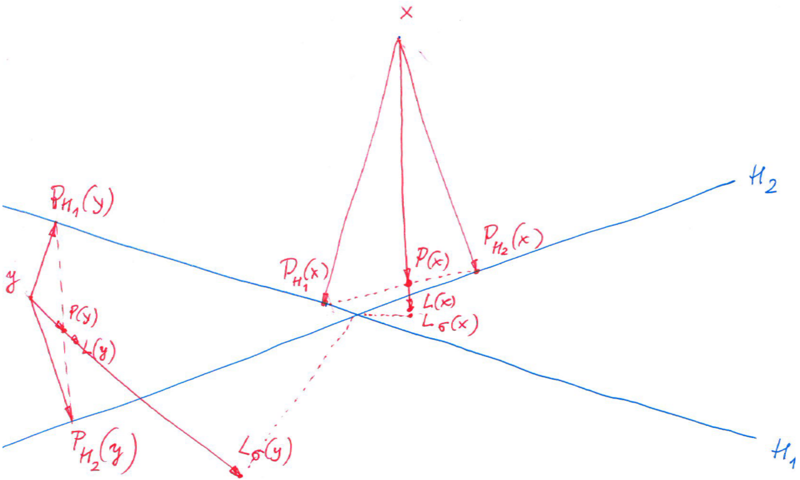
- $V_D x$ is a linearly regular cutter. Thus, for any $\lambda \in (0, 2)$ the method

$$x^{k+1} = U_{D,\lambda}(x^k)$$

converges linearly to a solution of $Ax \leq b$.

Extrapolated Landweber Operator for Linear Inequalities



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Thank you for your attention!